52BB

NATO Advanced Research Workshop — Spatio-Temporal Patterns in Nonequilibrium Complex Systems, Santa Fe, NM, April 13-17, 199 (in press).

Noise-Induced Transitions to Chaos

Michael Frey
Department of Mathematics, Bucknell University,
Lewisburg, PA 17837
Emil Simiu
Structures Division, Building and Fire Research Laboratory,
National Institute of Standards and Technology,
Gaithersburg, MD 20899

ABSTRACT

Additive noise is shown to induce chaotic motion with sensitive dependence on initial conditions in multistable dynamical systems. This establishes a fundamental connection between two fields hitherto viewed as distinct: deterministic chaos and stochastic differential equations modeling the dynamics of multistable systems. Our results for additive noise are then generalized to multiplicative noise. Using a newly introduced model of shot noise, these results for multiplicative noise are applied to the Duffing oscillator with shot noise-like dissipation.

INTRODUCTION

Multistable systems such as the Duffing oscillator can exhibit irregular (i.e., neither periodic nor quasiperiodic) motion with jumps. Such motion is referred to as basin-hopping [1] or stochastic chaos [2] when induced by noise, and deterministic chaos in the absence of noise. Deterministic and stochastic chaos have been viewed as distinct and have been analyzed from different, indeed contrasting, points of view.

In fact, for a wide class of systems stochastic and deterministic chaos can be not only indistinguishable phenomenologically but also closely related mathematically. We show this for the paradigmatic case of one-degree-of-freedom multistable systems whose unperturbed counterparts have homoclinic and heteroclinic orbits. (Extensions of the theory to higher-degree-of-freedom and spatially extended systems are underway.) When perturbed by weak damping and deterministic periodic forcing, the dynamics of these systems are periodic or quasiperiodic over certain regions of the system parameter space. Over other regions of parameter space the dynamics may be sensitively dependent on initial conditions; i.e., exhibit a topological equivalence to the Smale horseshoe map. We show that a transition from periodic or quasiperiodic motion to chaotic motion with sensitive dependence on initial conditions is possible through the introduction of noise.

We develop computable expressions providing 1) necessary conditions for the occurrence of stochastic chaos with jumps and 2) measures of chaotic transport characterizing the "intensity" of the chaos. We obtain these expressions using 1) the Melnikov transform and the notion of phase space flux, both developed for deterministic systems, and 2) uniformly bounded path approximations of Gaussian noise and shot noise.

The remainder of the paper is divided into three sections. The first section reviews results for systems perturbed by weak additive noise. In the second section we present results for multiplicatively perturbed systems. These results are used with a recently introduced model of shot noise to treat the Duffing oscillator with shot noise-like dissipation. The last section contains summary comments.

SYSTEMS WITH ADDITIVE EXCITATION

We consider the additively excited dynamical system

$$\ddot{x} = -V'(x) + \varepsilon[\gamma g(t) + \rho G(t) - \kappa \dot{x}] \tag{1}$$

where V is an energy potential, $0 < \varepsilon \ll 1$, and g and G represent deterministic and stochastic forcing functions, respectively. g is assumed to be bounded, $|g(t)| \leq 1$, and uniformly continuous (UC). The parameters ρ, γ and κ are nonnegative and fix the relative amounts of damping and external forcing in the model. The unperturbed ($\varepsilon = 0$) counterpart of (1) is assumed to have two hyperbolic fixed points connected by a heteroclinic orbit $\vec{x}_s = (x_s(t), \dot{x}_s(t))$. If the two hyperbolic fixed points coincide, then $\vec{x}_s$ is homoclinic.

Consider the random forcing

$$G(t) = \sqrt{\frac{2}{N}} \sum_{n=1}^{N} \frac{\sigma}{S(\nu_n)} \cos(\nu_n t + \varphi_n) \tag{2}$$

where $\{\nu_n, \varphi_n; n = 1, 2, \ldots, N\}$ are mutually independent random variables, $\{\nu_n; n = 1, 2, \ldots, N\}$ are nonnegative with common distribution Ψ_o, $\{\varphi_n; n = 1, 2, \ldots, N\}$ are identically uniformly distributed over the interval $[0, 2\pi]$ and N is a fixed parameter of the model. S and σ in (2) are defined below. The process G is a randomly weighted modification of the Shinozuka noise model [3].

Let $\mathcal{F}$ denote the linear filter with impulse response $h(t) = \dot{x}_s(-t)$ where $\dot{x}_s(t)$ is the velocity component of the orbit $\vec{x}_s$ of system (1). $\mathcal{F}$ is called the system orbit filter and its output is $\mathcal{F}[u] = u * h$ where $u = u(t)$ is the filter input and $u * h$ is the convolution of u and h. S in (2) is then defined to be the modulus $S(\nu) = |H(\nu)|$ of the orbit filter transfer function

$$H(\nu) = \int_{-\infty}^{\infty} h(t) e^{-j\nu t} dt$$

and σ in (2) is

$$\sigma^2 = \int_0^\infty S^2(\nu)\Psi(d\nu).$$

Let the distribution Ψ_o of the angular frequencies ν_n in (2) have the form

$$\Psi_o(A) = \frac{1}{\sigma^2}\int_A S^2(\nu)\Psi(d\nu)$$

where A is any Borel subset of $\mathcal{R}$. S is assumed to be bounded away from zero on the support of Ψ, $S(\nu) > S_m > 0$ a.e. Ψ. Under this condition S is said to be Ψ-admissible. If S is Ψ-admissible, then it is also bounded away from zero on the support of Ψ_o and $1/S(\nu_n) < 1/S_m$ a.s. Ψ_o. The following results for G and its filtered counterpart $\mathcal{F}[G]$ are proved in [4]: 1) the processes G and $\mathcal{F}[G]$ are each zero-mean and stationary; 2) if S is Ψ-admissible then G is uniformly bounded with $|G(t,\omega)| \leq \sqrt{2N/S_m}$ for all $t \in \mathcal{R}$ and $\omega \in \Omega$; 3) the marginal distribution of $\mathcal{F}[G]$ is that of the sum

$$\sigma\sqrt{\frac{2}{N}}\sum_{n=1}^{N}\cos U_n$$

where $\{U_n; n = 1, \ldots, N\}$ are independent random variables uniformly distributed on the interval $[0, 2\pi]$; 4) the processes G and $\mathcal{F}[G]$ are each asymptotically Gaussian in the limit as $N \to \infty$ and the random variables $G(t)$ and $\mathcal{F}[G](t)$ are, for each t, asymptotically Gaussian; 5) the spectrum of G is $2\pi\Psi$ and G has unit variance; 6) if the spectrum Ψ of G is continuous, then $\mathcal{F}[G]$ is ergodic; and 7) the spectrum of $\mathcal{F}[G]$ is $2\pi\Psi_o$ and its variance is σ^2. From this last result, it follows that a modified Shinozuka noise process can be constructed with any prescribed spectrum.

For sufficiently small ε, the hyperbolic fixed points of the unperturbed system are displaced to a nearby invariant manifold and the stable and unstable manifolds associated with the homoclinic or heteroclinic orbit of (1) separate [5]. The distance between the separated manifolds is expressible as an asymptotic expansion $\varepsilon M + O(\varepsilon^2)$ where M is a computable quantity called the Melnikov function. The separated manifolds may intersect transversely and, if such intersections occur, they are infinite in number and define lobes marking the transport of phase space [6]. The amount of phase space transported, the phase space flux, is a measure of the chaoticity of the dynamics [6]. For the case of small perturbations, the average phase space flux has the asymptotic expansion $\varepsilon\Phi + O(\varepsilon^2)$ [6] where Φ, here called the flux factor, is a time average of the positive part of the Melnikov function:

$$\Phi = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} M^+(\theta_1 - t, \theta_2 - t)dt. \tag{3}$$

To apply Melnikov theory to the random perturbation G, G must be uniformly bounded and uniformly continuous across both time and ensemble. The noise model

G in (2) is uniformly bounded as noted earlier. It is shown in [7] that G has the needed degree of continuity if and only if G is bandlimited.

The Melnikov function for system (1) is then given by the Melnikov transform $\mathcal{M}[g,G]$ of g and G [8]:

$$\begin{aligned} M(t_1,t_2) &= \mathcal{M}[g,G] \\ &= -\kappa\int_{-\infty}^{\infty}\dot{x}_s^2(t)dt+\gamma\int_{-\infty}^{\infty}\dot{x}_s(t)g(t+t_1)dt \\ &\qquad +\rho\int_{-\infty}^{\infty}\dot{x}_s(t)G(t+t_2)dt. \end{aligned} \tag{4}$$

Since $h(t)=\dot{x}_s(-t)$, denoting the integral of $\dot{x}_s^2$ by I, we obtain

$$M(t_1,t_2)=-I\kappa+\gamma\mathcal{F}[g](t_1)+\rho\mathcal{F}[G](t_2). \tag{5}$$

The expectation and variance of $M(t_1,t_2)$ are, respectively,

$$E[M(t_1,t_2)]=-I\kappa+\gamma\mathcal{F}[g](t), Var[M(t_1,t_2)]=\rho^2\sigma^2=\rho^2\int_0^{\infty}S^2(\nu)\Psi(d\nu).$$

$M(t_1,t_2)$ is, like G, a Gaussian process in the limit as $N\to\infty$ indicating that the presence of even vanishingly small noise causes the Melnikov function to have simple zeros. The state of the system is thus driven from one basin of attraction to that of the competing attractor. Such motion is interpretable as chaotic motion on a single strange attractor [6].

Substitute (5) into (3). Then

$$\Phi=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}[\gamma\mathcal{F}[g](\theta_1-s)+\rho\mathcal{F}[G](\theta_2-s)-I\kappa]^+ds. \tag{6}$$

Existence of the limit in (6) depends on the nature of the excitations g and G and their corresponding convolutions $\mathcal{F}[g]=g*h$ and $\mathcal{F}[G]=G*h$.

To ensure the existence of the limit in (6), we assume that g is asymptotic mean stationary (AMS): a stochastic process $X(t)$ is defined to be AMS if [9] the limits

$$\mu_X(A)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}E[1_A(X(t))]dt \tag{7}$$

exists for each real Borel set $A\in\mathcal{R}$. Here 1_A is the indicator function, $1_A(x)=1$ for $x\in A$ and $1_A(x)=0$ otherwise. This definition applies, in particular, to deterministic functions $X(t)$. We note that all periodic and quasiperiodic functions are AMS. If the limits in (7) exist then μ_X is a probability measure. μ_X is called the stationary mean (SM) distribution of the process X.

$\mathcal{F}$ is linear so its output $\mathcal{F}[g]$ is AMS and we denote the SM distribution of $\mathcal{F}[g]$ by $\mu_{\mathcal{F}[g]}$. Assume the spectrum of G is continuous. Then $\mathcal{F}[G]$ is ergodic.

Ergodicity implies asymptotic mean stationarity [9] so $\mathcal{F}[G]$ is AMS also with SM distribution $\mu_{\mathcal{F}[G]}$. All AMS deterministic functions are ergodic so $\mathcal{F}[g]$, like $\mathcal{F}[G]$, is ergodic. Inasmuch as $\mathcal{F}[g]$ is deterministic, $\mathcal{F}[g]$ and $\mathcal{F}[G]$ are jointly ergodic with SM distribution $\mu_{\mathcal{F}[g]} \times \mu_{\mathcal{F}[G]}$ [4]. Then the limit (6) exists and can be expressed in terms of the SM distributions $\mu_{\mathcal{F}[g]}$ and $\mu_{\mathcal{F}[G]}$. These observations are the basis for the following result.

Theorem 1 [4, 7]: Suppose g is AMS and G is a Ψ-admissible modified Shinozuka process with continuous bandlimited spectrum. Then the flux factor Φ is approximately

$$\Phi \doteq E[(\gamma A + \rho\sigma Z - I\kappa)^+]$$

where Z is a standard Gaussian random variable. The error in this approximation decreases to zero as $N \to \infty$.

SYSTEMS WITH MULTIPLICATIVE EXCITATION

We now generalize the simple additive model used in (1) to the multiplicative excitation model:

$$\gamma(x, \dot{x})g(t) + \rho(x, \dot{x})G(t). \tag{8}$$

As in the additive excitation model, the function g represents deterministic forcing while G is a stochastic process representing random forcing.

The Melnikov function is calculated as in (4) to be

$$M(t_1, t_2) = \mathcal{M}[g, G] = \int_{-\infty}^{\infty} \dot{x}_s(t)[\gamma(x_s(t), \dot{x}_s(t))g(t + t_1) + \rho(x_s(t), \dot{x}_s(t))G(t + t_2)]dt.$$

We define orbit filters $\mathcal{F}_1$ and $\mathcal{F}_2$ with impulse responses

$$h_1(t) = \dot{x}_s(-t)\gamma(x_s(-t), \dot{x}_s(-t)), \quad h_2(t) = \dot{x}_s(-t)\rho(x_s(-t), \dot{x}_s(-t))$$

and corresponding transfer functions $H_1(\nu)$ and $H_2(\nu)$. Then

$$M(t_1, t_2) = \mathcal{F}_1[g](t_1) + \mathcal{F}_2[G](t_2). \tag{9}$$

We see that the orbit filter $\mathcal{F}$ in the additive model is replaced in the multiplicative model by two different orbit filters $\mathcal{F}_1$ and $\mathcal{F}_2$ and that the filters $\mathcal{F}_1$ and $\mathcal{F}_2$ are linear, time-invariant and noncausal with impulse responses given solely in terms of the orbit $\vec{x}_s$ of the unperturbed system and the functions γ and ρ.

Substituting (9) into (3) gives

$$\Phi = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [\rho\mathcal{F}_1[g](\theta_1 - s) + \gamma\mathcal{F}_2[G](\theta_2 - s)]^+ ds. \tag{10}$$

Just as in the case of the additive excitation model, existence of the limit in (10) hinges on the joint ergodicity of the function $\mathcal{F}_1[g] = g * h_1$ and the process $\mathcal{F}_2[G] = G * h_2$.

Theorem 2: Consider system (1) but with the multiplicative excitation model in (8) such that g is AMS and $\mathcal{F}_2[G]$ is ergodic. Let $\mu_{\mathcal{F}_1[g]}$ and $\mu_{\mathcal{F}_2[G]}$ be the SM distributions of $\mathcal{F}_1[g]$ and $\mathcal{F}_2[G]$, respectively. Then the limit in (10) exists, the flux factor Φ is nonrandom and

$$\Phi = E[(\gamma A + \rho B)^+]$$

where A is a random variable with distribution $\mu_{\mathcal{F}_1[g]}$, B is a random variable with distribution $\mu_{\mathcal{F}_2[G]}$ and A and B are independent.

As an example of a system with multiplicative shot noise, we consider the Duffing oscillator with weak forcing and non-autonomous damping:

$$\ddot{x} = x - x^3 + \varepsilon[\gamma g(t) - \kappa(K_N(t) + \eta)\dot{x}]. \tag{11}$$

Here $\gamma \geq 0$, $\kappa \geq 0$ and $\eta \geq 0$ are constants, g is deterministic and bounded $|g(t)| \leq 1$, and K_N is a form of shot noise. The perturbation in (11) is a particular case of the multiplicative excitation model (8) with $\gamma(x,\dot{x}) = \gamma$, $\rho(x,\dot{x}) = -\kappa\dot{x}$, and $G(t) = K_N(t)+\eta$. $\kappa(K_N(t)+\eta)$ in (11) serves as a time-varying damping factor and plays the same role as the constant κ in (1). The two terms $\kappa\eta$ and κK_N represent, respectively, viscous and shot noise-like damping forces. We assume for this example that $\eta = 0$. The shot response r (see below) of K_N is assumed to be nonnegative so that the factor κK_N is nonnegative.

The usual model of constant-rate shot noise is a stochastic process of the form [10]

$$K(t) = \sum_{k \in \mathcal{Z}} r(t - T_k) \tag{12}$$

where $\mathcal{Z}$ is the set of integers, $\{T_k, k \in \mathcal{Z}\}$ are the epochs (shots) of a Poisson process with rate $\lambda > 0$ and r, the shot response of the process K, is bounded and square-integrable.

The shot noise model K in (12) is neither bounded nor EUC and cannot be used in conjunction with Melnikov theory in calculating the phase space flux in chaotic systems. A modification of the model which approximates K and yet has the requisite path properties has been developed [7]:

$$K_N(t) = \sum_{j \in \mathcal{Z}} \sum_{k=1}^{2^N} r(t - T_{jkN} - A_j - T) \tag{13}$$

where N is a positive integer, $A_j = 2^N(j - 1/2)/\lambda$ and $\{T, T_{jkN}, j \in \mathcal{Z}, k = 1, \ldots, 2^N\}$ are independent random variables such that for each N and j, $\{T_{jkN}, k = 1, 2, \ldots, 2^N\}$ are identically uniformly distributed in the interval $(A_j, A_{j+1}]$ and T is uniformly distributed between 0 and $2^N/\lambda$. λ is again the rate of the process; it is the mean number of epochs (shots) T_{jkN} per unit time. We assume just as for K, that r in (13) is bounded and square-integrable, that r is UC and that the radial majorant

$$r^*(t) = \sup_{|\tau| \geq |t|} |r(\tau)|$$

of the shot response is integrable. According to this specification of K_N, realizations of the process are obtained by partitioning the real line into the intervals $(A_j, A_{j+1}]$ of length $2^N/\lambda$ with common random phase T and then placing 2^N epochs independently and at random in each interval. The random phase T eliminates the (ensemble) cyclic nonstationarity produced by partitioning by $(A_j, A_{j+1}]$. It can be shown [7] that for large N the shot noise K_N closely approximates the standard shot noise model K in all important respects. Also, K_N, unlike K, can be used in Melnikov's method-type calculations of the flux factor.

According to Theorem 2, the Melnikov function for the Duffing oscillator (11) is

$$M(t_1, t_2) = \mathcal{F}_1[g](t_1) + \mathcal{F}_2[G](t_2)$$

where

$$h_1(t) = \gamma\sqrt{2}\operatorname{sech}t \tanh t$$

and

$$h_2(t) = -2\kappa\operatorname{sech}^2 t \tanh^2 t.$$

The corresponding moduli of the filters $\mathcal{F}_1$ and $\mathcal{F}_2$ are

$$S_1(\nu) = \sqrt{2}\pi\gamma\nu\operatorname{sech}\frac{\pi\nu}{2}$$

and

$$S_2(\nu) = 4\kappa\int_0^\infty \operatorname{sech}^2 t \tanh^2 t \cos\nu t dt.$$

We have $S_1(0) = 0$ so the d.c. component (if any) of g is completely removed by $\mathcal{F}_1$ and has no effect on the Melnikov function. K_N does have a d.c. component; K_N is ergodic so its d.c. component is $E[K_N] = \lambda R(0)$ where

$$R(0) = \int_{-\infty}^{\infty} r(t)dt > 0.$$

$S_2(0) = 4\kappa/3 > 0$ so the d.c. component of K_N passed by $\mathcal{F}_2$ is

$$E[\mathcal{F}_2[K_N]] = E[K_N]S_2(0) = \frac{4\kappa\lambda R(0)}{3}.$$

The presence of a d.c. component plays a pivotal role in shifting the parametric threshold for chaos. See [4] for further discussion.

Assume the deterministic forcing function g is AMS. K_N is uniformly bounded and EUC and $\mathcal{F}_2[K_N]$ is ergodic. Thus $\mathcal{F}_1[g]$ and $\mathcal{F}_2[K_N]$ are jointly ergodic. By Theorem 2, the flux factor Φ exists and

$$\Phi = E[(A - B_N)^+]$$

where the distribution of A is $\mu_{\mathcal{F}_1[g]}$, the distribution of B_N is $\mu_{\mathcal{F}_2[K_N]}$ and A and B_N are independent.

We noted earlier that the distribution of $\mathcal{F}_2[K_N]$ is, for large N, approximately that of the shot noise $\mathcal{F}_2[K]$. This is the basis for the following theorem.

Theorem 3: The flux factor Φ for the Duffing oscillator (11) with weak forcing and shot noise damping coefficent κK_N is approximately

$$\Phi \doteq E[(A-B)^+]$$

where A is $\mu_{\mathcal{F}_1[g]}$-distributed, B is $\mu_{\mathcal{F}_2[K]}$-distributed, A and B are independent and K is the shot noise (12). The error in this approximation decreases to zero as $N \to \infty$.

Φ can be calculated numerically as follows for given system parameters ν, γ and κ and shot parameters λ and r. Define

$$\Phi' = \frac{\Phi}{\gamma S_1(\nu)}, \quad A' = \frac{A}{\gamma S_1(\nu)}, \quad B' = \frac{B_N}{\gamma S_1(\nu)}, \quad \lambda' = \frac{16\lambda R^2(0)}{9J}, \quad \kappa' = \frac{3\kappa J}{4\gamma S_1(\nu) R(0)}$$

where

$$J = \int_{-\infty}^{\infty} (r * h)^2(t)dt.$$

Then

$$\Phi' = E[(A'-B')^+].$$

The random variable B' is approximately gamma-distributed [11] with density

$$\frac{t^{\alpha-1}e^{-t/\beta}}{\beta^\alpha \Gamma(\alpha)}, \quad t > 0$$

where the parameters α and β are determined by the condition that $E[B']$ and $Var[B']$ equal the mean and the variance, respectively, of the gamma distribution. $\Phi' = \Phi'(\kappa', \lambda')$ is plotted in Figure 1.

SUMMARY

Noise can cause multistable dynamical systems to exhibit chaotic motion with sensitive dependence on initial conditions. The theory applicable to noise-induced chaotic dynamics reviewed in this paper rests primarily on the concept of the Melnikov transform and on techniques for approximating noise with any given spectrum and marginal distribution by uniformly bounded, ensemble uniformly continuous processes. The results described here apply to weakly perturbed, one-degree-of-freedom dynamical systems featuring homoclinic or heteroclinic orbits. Results were first given for additive perturbation and then generalized to multiplicative perturbation. Extensions of

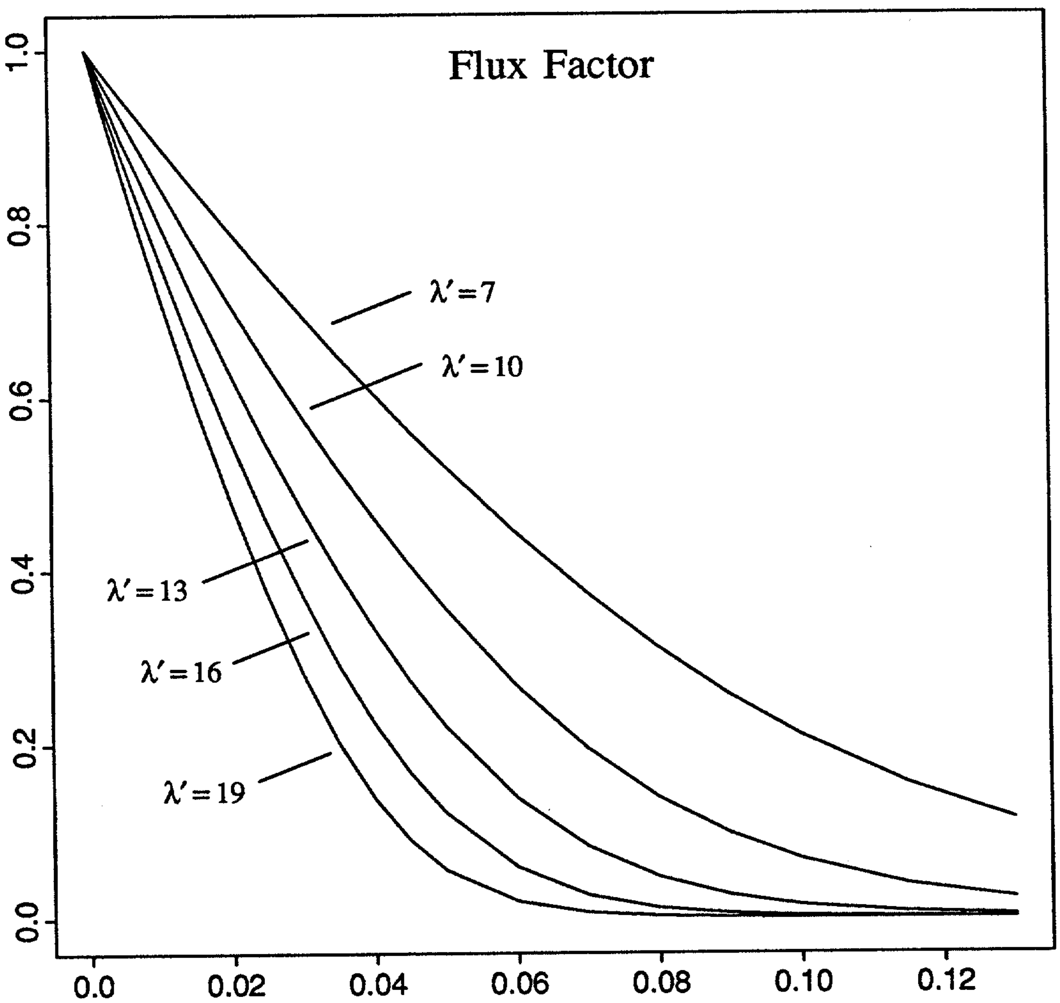

Figure 1. The flux factor Φ' as a function of the damping constant κ' for various shot rates λ'.

this work to higher-degree-of-freedom systems and to spatially extended systems are in progress.

ACKNOWLEDGEMENT

This research was supported in part by the Ocean Engineering Division of the Office of Naval Research, Grant nos. N-00014-93-1-0248 and N-00014-93-F-0028.

REFERENCES

[1] Arecchi, F.T., Badii, R. and Politi, A., Generalized multistability and noise-induced jumps in a nonlinear dynamical system, Phys. Rev. A, Vol.32, No.1, pp. 402-408, 1985.

[2] Arneodo, A., Argoul, A., Elzegaray, J. and Richetti, P., Homoclinic Chaos in Chemical Systems, Physica D, Vol. 62, pp. 134-169, 1993.

[3] Shinozuka, M. and Jan, C.-M., Digital Simulation of Random Processes and its Applications, Jour. of Sound and Vibration, Vol. 25, No. 1, pp. 111-128, 1972.

[4] Frey, M. and Simiu, E., Noise-induced chaos and phase space flux, Physica D, Vol. 63, pp. 321-340, 1993.

[5] Arrowsmith, D.K. and Place, C.M., An Introduction to Dynamical Systems, Cambridge University Press, New York, 1990.

[6] Wiggins, S., Chaotic Transport in Dynamical Systems, Springer-Verlag, New York, 1991.

[7] Frey, M. and Simiu, E., "Deterministic and Stochastic Chaos," in Computational Stochastic Mechanics, Cheng, A.H-D. and Yang, C.Y. (eds.), Elsevier Applied Science, London, June 1993.

[8] Meyer, K.R. and Sell, G.R., Melnikov Transforms, Bernoulli Bundles, and Almost Periodic Perturbations, Trans. Am. Math. Soc., Vol.314, No.1, 1989.

[9] Gray, R.M., Probability, Random Processes and Ergodic Properties, Springer-Verlag, New York, 1988.

[10] Snyder, D. and Miller, M., Random Point Processes in Time and Space, Springer-Verlag, New York, 1991.

[11] Papoulis, A., The Fourier Integral and its Applications, McGraw-Hill Book Co., New York, 1962.